TWISTED CONVOLUTION, PSEUDO-DIFFERENTIAL OPERATORS AND FOURIER MODULATION SPACES

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ABSTRACT. We discuss continuity of the twisted convolution on (weighted) Fourier modulation spaces. We use these results to establish continuity results for the twisted convolution on Lebesgue spaces. For example we prove that if ω is an appropriate weight and $1 \leq p \leq 2$, then $L^p_{(\omega)}$ is an algebra under the twisted convolution.

We also discuss continuity for pseudo-differential operators with symbols in Fourier modulation spaces.

0. INTRODUCTION

In this paper we continue the discussions from [22] concerning continuity and algebraic properties for pseudo-differential operators in background of Lebesgue spaces and the theory of modulation spaces. These investigations also involve studies of twisted convolutions, which are connected to the pseudo-differential calculus, in the sense that the Fourier transform of a Weyl product is essentially a twisted convolution of the Weyl symbols. By combining the latter property with continuity results of the Weyl product on modulation spaces in [22], we establish continuity properties of the twisted convolution on Fourier modulation spaces. From these results we thereafter prove continuity properties of the twisted convolution on weighted Fourier Lebesgue spaces.

We also consider continuity for pseudo-differential operators with symbols in Fourier modulation spaces. We establish continuity properties of such pseudo-differential operators when acting between modulation spaces and Fourier modulation spaces. These investigations are based on an important result by Cordero and Okoudjou in [2] concerning mapping properties of short-time Fourier transforms on modulation spaces.

The (classical) modulation spaces $M^{p,q}$, $p, q \in [1, \infty]$, as introduced by Feichtinger in [5], consist of all tempered distributions whose short-time Fourier transforms (STFT) have finite mixed $L^{p,q}$ norm. (Cf. [8] and the references therein for an updated describtion modulation spaces.) It follows that the parameters p and q to some extent quantify the degrees of asymptotic decay and singularity of the distributions in $M^{p,q}$. The theory of modulation spaces was developed further and generalized in [6, 7, 9–11, 14], where Feichtinger and Gröchenig established the theory of coorbit spaces. In particular, the modulation

spaces $M^{p,q}_{(\omega)}$ and $W^{p,q}_{(\omega)}$, where ω denotes a weight function on phase (or time-frequency shift) space, appears as the set of tempered (ultra-) distributions whose STFT belong to the weighted and mixed Lebesgue space $L^{p,q}_{1,(\omega)}$ and $L^{p,q}_{2,(\omega)}$ respectively. (See Section 1 for strict definitions.) By choosing the weight ω in appropriate ways, the space $W^{p,q}_{(\omega)}$ becomes a Wiener amalgam space, introduced in [3] by Feichtinger.

A major idea behind the design of these spaces was to find useful Banach spaces, which are defined in a way similar to Besov and Triebel-Lizorkin spaces, in the sense of replacing the dyadic decomposition on the Fourier transform side, characteristic to Besov and Triebel-Lizorkin spaces, with a *uniform* decomposition. From the construction of these spaces, it turns out that modulation spaces of the form $M^{p,q}_{(\omega)}$ and Besov spaces in some sense are rather similar, and sharp embeddings between these spaces can be found in [36,38], which are improvements of certain embeddings in [13]. (See also [28] for verification of the sharpness.) In the same way it follows that modulation spaces of the form $W^{p,q}_{(\omega)}$ and Triebel-Lizorkin spaces are rather similar.

During the last 15 years many results have been proved which confirm the usefulness of the modulation spaces and their Fourier transforms in time-frequency analysis, where they occur naturally. For example, in [11, 15, 20], it is shown that all such spaces admit reconstructible sequence space representations using Gabor frames.

Parallel to this development, modulation spaces have been incorporated into the calculus of pseudo-differential operators. In fact, in [27], Sjöstrand introduced the modulation space $M^{\infty,1}$, which contains non-smooth functions, as a symbol class and proved that $M^{\infty,1}$ corresponds to an algebra of operators which are bounded on L^2.

Gröchenig and Heil thereafter proved in [15, 17] that each operator with symbol in $M^{\infty,1}$ is continuous on all modulation spaces $M^{p,q}$, $p,q \in [1,\infty]$. This extends Sjöstrand's result since $M^{2,2} = L^2$. Some generalizations to operators with symbols in general unweighted modulation spaces were obtained in [18,36], and in [37,39] some further extensions involving weighted modulation spaces are presented. Modulation spaces in pseudodifferential calculus is currently an active field of research (see e. g. [16, 18, 19, 21, 24–26, 28–30, 33, 36, 39]).

In the Weyl calculus of pseudo-differential operators, operator composition corresponds on the symbol level to the Weyl product, sometimes also called the twisted product, denoted by #. A problem in this field is to find conditions on the weight functions ω_j and $p_j, q_j \in [1,\infty]$, that are necessary and sufficient for the map

$$(0.1) \qquad \mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d}) \ni (a_1, a_2) \mapsto a_1 \# a_2 \in \mathscr{S}(\mathbf{R}^{2d})$$

to be uniquely extendable to a map from $M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $M^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, which is continuous in the sense that for some constant

$C > 0$ it holds

$$(0.2) \qquad \|a_1 \# a_2\|_{M^{p_0,q_0}_{(\omega_0)}} \leq C\|a_1\|_{M^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{M^{p_2,q_2}_{(\omega_2)}},$$

when $a_1 \in M^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in M^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$. Important contributions in this context can be found in $[16, 22, 24, 27, 33]$, where Theorem $0.3'$ in $[22]$ seems to be the most general result so far.

The Weyl product on the Fourier transform side is given by a twisted convolution, $*_\sigma$. It follows that the continuity questions here above are the same as finding appropriate conditions on ω_j and $p_j, q_j \in [1, \infty]$, in order for the map

$$(0.3) \qquad \mathscr{S}(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d}) \ni (a_1, a_2) \mapsto a_1 *_\sigma a_2 \in \mathscr{S}(\mathbf{R}^{2d})$$

to be uniquely extendable to a map from $W^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times W^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ to $W^{p_0,q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, which is continuous in the sense that for some constant $C > 0$ it holds

$$(0.4) \qquad \|a_1 *_\sigma a_2\|_{W^{p_0,q_0}_{(\omega_0)}} \leq C\|a_1\|_{W^{p_1,q_1}_{(\omega_1)}} \|a_2\|_{W^{p_2,q_2}_{(\omega_2)}},$$

when $a_1 \in W^{p_1,q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in W^{p_2,q_2}_{(\omega_2)}(\mathbf{R}^{2d})$. In this context the continuity result which corresponds to Theorem $0.3'$ in $[22]$ is Theorem 2.3 in Section 2.

In the end of Section 2 we especially consider the case when $p_j = q_j = 2$. In this case, $W^{2,2}_{(\omega_j)}$ agrees with $L^2_{(\omega_j)}$, for appropriate choices of ω_j. Hence, for such ω_j, it follows immediately from Theorem 2.3 that the map (0.3) extends to a continuous mapping from $L^2_{(\omega_1)}(\mathbf{R}^{2d}) \times L^2_{(\omega_2)}(\mathbf{R}^{2d})$ to $L^2_{(\omega_0)}(\mathbf{R}^{2d})$, and that

$$\|a_1 *_\sigma a_2\|_{L^2_{(\omega_0)}} \leq C\|a_1\|_{L^2_{(\omega_1)}} \|a_2\|_{L^2_{(\omega_2)}},$$

when $a_1 \in L^2_{(\omega_1)}(\mathbf{R}^{2d})$ and $a_2 \in L^2_{(\omega_2)}(\mathbf{R}^{2d})$. In Section 2 we prove a more general result, by combining this result with Young's inequality, and then using interpolation. Finally we use these results in Section 3 to enlarge the class of possible window functions in the definition of modulation space norm.

1. Preliminaries

In this section we recall some notations and basic results. The proofs are in general omitted.

We start by discussing appropriate conditions for the involved weight functions. Assume that ω and v are positive and measureable functions on $\mathbf{R}^d$. Then ω is called v-moderate if

$$(1.1) \qquad \omega(x + y) \leq C\omega(x)v(y)$$

for some constant C which is independent of $x, y \in \mathbf{R}^d$. If v in (1.1) can be chosen as a polynomial, then ω is called polynomially moderated. We let $\mathscr{P}(\mathbf{R}^d)$ be the set of all polynomially moderated functions on

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$\mathbf{R}^d$. If $\omega(x,\xi) \in \mathscr{P}(\mathbf{R}^{2d})$ is constant with respect to the x-variable (ξ-variable), then we sometimes write $\omega(\xi)$ ($\omega(x)$) instead of $\omega(x,\xi)$. In this case we consider ω as an element in $\mathscr{P}(\mathbf{R}^{2d})$ or in $\mathscr{P}(\mathbf{R}^d)$ depending on the situation.

The Fourier transform $\mathscr{F}$ is the linear and continuous mapping on $\mathscr{S}'(\mathbf{R}^d)$ which takes the form

$$(\mathscr{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-d/2} \int_{\mathbf{R}^d} f(x)e^{-i\langle x,\xi\rangle}\,dx$$

when $f \in L^1(\mathbf{R}^d)$. We recall that $\mathscr{F}$ is a homeomorphism on $\mathscr{S}'(\mathbf{R}^d)$ which restricts to a homeomorphism on $\mathscr{S}(\mathbf{R}^d)$ and to a unitary operator on $L^2(\mathbf{R}^d)$.

Let $\varphi \in \mathscr{S}'(\mathbf{R}^d)$ be fixed, and let $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the short-time Fourier transform $V_\varphi f(x,\xi)$ of f with respect to the *window function* φ is the tempered distribution on $\mathbf{R}^{2d}$ which is defined by

$$V_\varphi f(x,\xi) \equiv \mathscr{F}(f\,\overline{\varphi(\cdot - x)})(\xi).$$

If $f, \varphi \in \mathscr{S}(\mathbf{R}^d)$, then it follows that

$$V_\varphi f(x,\xi) = (2\pi)^{-d/2} \int f(y)\overline{\varphi(y-x)}e^{-i\langle y,\xi\rangle}\,dy.$$

Next we recall some properties on modulation spaces and their Fourier transforms. Assume that $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and that $p,q \in [1,\infty]$. Then the mixed Lebesgue space $L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{1,(\omega)}} < \infty$, and $L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$ consists of all $F \in L^1_{loc}(\mathbf{R}^{2d})$ such that $\|F\|_{L^{p,q}_{2,(\omega)}} < \infty$. Here

$$\|F\|_{L^{p,q}_{1,(\omega)}} = \left(\int \left(\int |F(x,\xi)\omega(x,\xi)|^p\,dx \right)^{q/p} d\xi \right)^{1/q},$$

and

$$\|F\|_{L^{p,q}_{2,(\omega)}} = \left(\int \left(\int |F(x,\xi)\omega(x,\xi)|^q\,d\xi \right)^{p/q} dx \right)^{1/p},$$

with obvious modifications when $p = \infty$ or $q = \infty$.

Assume that $p,q \in [1,\infty]$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$ and $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ are fixed. Then the *modulation space* $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the Banach space which consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

(1.2) $$\|f\|_{M^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{1,(\omega)}} < \infty.$$

The modulation space $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ is the Banach space which consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

(1.3) $$\|f\|_{W^{p,q}_{(\omega)}} \equiv \|V_\varphi f\|_{L^{p,q}_{2,(\omega)}} < \infty.$$

The definitions of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and $W^{p,q}_{(\omega)}(\mathbf{R}^d)$ are independent of the choice of φ and different φ gives rise to equivalent norms. (See Proposition 1.1 below). From the fact that

$$V_{\widehat{\varphi}}\widehat{f}(\xi, -x) = e^{i\langle x, \xi \rangle} V_{\check{\varphi}} f(x, \xi), \qquad \check{\varphi}(x) = \varphi(-x),$$

it follows that

$$f \in W^{p,q}_{(\omega)}(\mathbf{R}^d) \quad \Longleftrightarrow \quad \widehat{f} \in M^{p,q}_{(\omega_0)}(\mathbf{R}^d), \qquad \omega_0(\xi, -x) = \omega(x, \xi).$$

For conveniency we set $M^p_{(\omega)} = M^{p,p}_{(\omega)}$, which agrees with $W^p_{(\omega)} = W^{p,p}_{(\omega)}$. Furthermore we set $M^{p,q} = M^{p,q}_{(\omega)}$ and $W^{p,q} = W^{p,q}_{(\omega)}$ if $\omega \equiv 1$. If ω is given by $\omega(x, \xi) = \omega_1(x)\omega_2(\xi)$, for some $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^d)$, then $W^{p,q}_{(\omega)}$ is a Wiener amalgam space, introduced by Feichtinger in [3].

The proof of the following proposition is omitted, since the results can be found in [4, 5, 9–11, 15, 36–39]. Here and in what follows, $p' \in [1, \infty]$ denotes the conjugate exponent of $p \in [1, \infty]$, i.e. $1/p + 1/p' = 1$ should be fulfilled.

Proposition 1.1. *Assume that $p, q, p_j, q_j \in [1, \infty]$ for $j = 1, 2$, and $\omega, \omega_1, \omega_2, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate and $\omega_2 \leq C\omega_1$ for some constant $C > 0$. Then the following are true:*

(1) *if $\varphi \in M^1_{(v)}(\mathbf{R}^d) \backslash 0$, then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if (1.2) holds, i.e. $M^{p,q}_{(\omega)}(\mathbf{R}^d)$. Moreover, $M^{p,q}_{(\omega)}$ is a Banach space under the norm in (1.2) and different choices of φ give rise to equivalent norms;*

(2) *if $p_1 \leq p_2$ and $q_1 \leq q_2$ then*

$$\mathscr{S}(\mathbf{R}^d) \hookrightarrow M^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^n) \hookrightarrow M^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^d) \hookrightarrow \mathscr{S}'(\mathbf{R}^d);$$

(3) *the L^2 product $(\,\cdot\,, \cdot\,)$ on $\mathscr{S}$ extends to a continuous map from $M^{p,q}_{(\omega)}(\mathbf{R}^n) \times M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$ to $\mathbf{C}$. On the other hand, if $\|a\| = \sup |(a, b)|$, where the supremum is taken over all $b \in \mathscr{S}(\mathbf{R}^d)$ such that $\|b\|_{M^{p',q'}_{(1/\omega)}} \leq 1$, then $\|\cdot\|$ and $\|\cdot\|_{M^{p,q}_{(\omega)}}$ are equivalent norms;*

(4) *if $p, q < \infty$, then $\mathscr{S}(\mathbf{R}^d)$ is dense in $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ and the dual space of $M^{p,q}_{(\omega)}(\mathbf{R}^d)$ can be identified with $M^{p',q'}_{(1/\omega)}(\mathbf{R}^d)$, through the form $(\,\cdot\,, \cdot\,)_{L^2}$. Moreover, $\mathscr{S}(\mathbf{R}^d)$ is weakly dense in $M^\infty_{(\omega)}(\mathbf{R}^d)$.*

Similar facts hold if the $M^{p,q}_{(\omega)}$ spaces are replaced by $W^{p,q}_{(\omega)}$ spaces.

Proposition 1.1 (1) allows us be rather vague concerning the choice of $\varphi \in M^1_{(v)} \backslash 0$ in (1.2) and (1.3). For example, if $C > 0$ is a constant and $\mathscr{A}$ is a subset of $\mathscr{S}'$, then $\|a\|_{W^{p,q}_{(\omega)}} \leq C$ for every $a \in \mathscr{A}$, means that the inequality holds for some choice of $\varphi \in M^1_{(v)} \backslash 0$ and every $a \in \mathscr{A}$. Evidently, a similar inequality is true for any other choice of $\varphi \in M^1_{(v)} \backslash 0$, with a suitable constant, larger than C if necessary.

In the following remark we list some other properties for modulation spaces. Here and in what follows we let $\langle x \rangle = (1+|x|^2)^{1/2}$, when $x \in \mathbf{R}^d$.

Remark 1.2. Assume that $p, p_1, p_2, q, q_1, q_2 \in [1, \infty]$ are such that

$$q_1 \leq \min(p, p'), \quad q_2 \geq \max(p, p'), \quad p_1 \leq \min(q, q'), \quad p_2 \geq \max(q, q'),$$

and that $\omega, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that ω is v-moderate. Then the following is true:

(1) if $p \leq q$, then $W_{(\omega)}^{p,q}(\mathbf{R}^d) \subseteq M_{(\omega)}^{p,q}(\mathbf{R}^d)$, and if $p \geq q$, then $M_{(\omega)}^{p,q}(\mathbf{R}^d) \subseteq W_{(\omega)}^{p,q}(\mathbf{R}^d)$. Furthermore, if $\omega(x, \xi) = \omega(x)$, then

$$M_{(\omega)}^{p,q_1}(\mathbf{R}^d) \subseteq W_{(\omega)}^{p,q_1}(\mathbf{R}^d) \subseteq L_{(\omega)}^{p}(\mathbf{R}^d) \subseteq W_{(\omega)}^{p,q_2}(\mathbf{R}^d) \subseteq M_{(\omega)}^{p,q_2}(\mathbf{R}^d).$$

In particular, $M_{(\omega)}^2 = W_{(\omega)}^2 = L_{(\omega)}^2$. If instead $\omega(x, \xi) = \omega(\xi)$, then

$$W_{(\omega)}^{p_1,q}(\mathbf{R}^d) \subseteq M_{(\omega)}^{p,q_1}(\mathbf{R}^d) \subseteq \mathscr{F}L_{(\omega)}^{q}(\mathbf{R}^d) \subseteq M_{(\omega)}^{p_2,q}(\mathbf{R}^d) \subseteq W_{(\omega)}^{p_2,q}(\mathbf{R}^d).$$

Here $\mathscr{F}L_{(\omega_0)}^{q}(\mathbf{R}^d)$ consists of all $f \in \mathscr{S}'(\mathbf{R}^d)$ such that

$$\|\widehat{f}\,\omega_0\|_{L^q} < \infty;$$

(2) if $\omega(x, \xi) = \omega(x)$, then the following conditions are equivalent:
  - $M_{(\omega)}^{p,q}(\mathbf{R}^d) \subseteq C(\mathbf{R}^d)$;
  - $W_{(\omega)}^{p,q}(\mathbf{R}^d) \subseteq C(\mathbf{R}^d)$;
  - $q = 1$.

(3) $M^{1,\infty}(\mathbf{R}^d)$ and $W^{1,\infty}(\mathbf{R}^d)$ are convolution algebras. If $C_B'(\mathbf{R}^d)$ is the set of all measures on $\mathbf{R}^d$ with bounded mass, then

$$W^{1,\infty}(\mathbf{R}^d) \subseteq C_B'(\mathbf{R}^d) \subseteq M^{1,\infty}(\mathbf{R}^d);$$

(4) if $x_0 \in \mathbf{R}^d$ is fixed and $\omega_0(\xi) = \omega(x_0, \xi)$, then

$$M_{(\omega)}^{p,q} \cap \mathscr{E}' = W_{(\omega)}^{p,q} \cap \mathscr{E}' = \mathscr{F}L_{(\omega_0)}^{q} \cap \mathscr{E}';$$

(5) if $\omega(x, \xi) = \omega_0(\xi, -x)$, then the Fourier transform on $\mathscr{S}'(\mathbf{R}^d)$ restricts to a homeomorphism from $M_{(\omega)}^{p}(\mathbf{R}^d)$ to $M_{(\omega_0)}^{p}(\mathbf{R}^d)$. In particular, if $\omega = \omega_0$, then $M_{(\omega)}^{p}$ is invariant under the Fourier transform. Similar facts hold for partial Fourier transforms;

(6) for each $x, \xi \in \mathbf{R}^d$ we have

$$\|e^{i\langle \cdot, \xi \rangle} f(\cdot - x)\|_{M_{(\omega)}^{p,q}} \leq Cv(x, \xi)\|f\|_{M_{(\omega)}^{p,q}},$$

and

$$\|e^{i\langle \cdot, \xi \rangle} f(\cdot - x)\|_{W_{(\omega)}^{p,q}} \leq Cv(x, \xi)\|f\|_{W_{(\omega)}^{p,q}}$$

for some constant C which is independent of $f \in \mathscr{S}'(\mathbf{R}^d)$;

(7) if $\tilde{\omega}(x, \xi) = \omega(x, -\xi)$ then $f \in M_{(\omega)}^{p,q}$ if and only if $\overline{f} \in M_{(\tilde{\omega})}^{p,q}$;

(8) if $s \in \mathbf{R}$ and $\omega(x, \xi) = \langle \xi \rangle^s$, then $M_{(\omega)}^2 = W_{(\omega)}^2$ agrees with H_s^2, the Sobolev space of distributions with s derivatives in L^2. That is, H_s^2 consists of all $f \in \mathscr{S}'$ such that $\mathscr{F}^{-1}(\langle \, \cdot \, \rangle^s \widehat{f}) \in L^2$.

(See e. g. [4, 5, 9–11, 15, 36–39].)

We also need some facts in Section 2 in [39] on narrow convergence. For any $f \in \mathscr{S}'(\mathbf{R}^d)$, $\omega \in \mathscr{P}(\mathbf{R}^{2d})$, $\varphi \in \mathscr{S}(\mathbf{R}^d)$ and $p \in [1, \infty]$, we set

$$H_{f,\omega,p}(\xi) = \Big(\int_{\mathbf{R}^d} |V_\varphi f(x, \xi)\omega(x, \xi)|^p \, dx \Big)^{1/p}.$$

Definition 1.3. Assume that $f, f_j \in M_{(\omega)}^{p,q}(\mathbf{R}^d)$, $j = 1, 2, \dots$. Then f_j is said to converge *narrowly* to f (with respect to $p, q \in [1, \infty]$, $\varphi \in \mathscr{S}(\mathbf{R}^d) \setminus 0$ and $\omega \in \mathscr{P}(\mathbf{R}^{2d})$), if the following conditions are satisfied:

(1) $f_j \to f$ in $\mathscr{S}'(\mathbf{R}^d)$ as j turns to ∞;
(2) $H_{f_j,\omega,p}(\xi) \to H_{f,\omega,p}(\xi)$ in $L^q(\mathbf{R}^d)$ as j turns to ∞.

Remark 1.4. Assume that $f, f_1, f_2, \dots \in \mathscr{S}'(\mathbf{R}^d)$ satisfies (1) in Definition 1.3, and assume that $\xi \in \mathbf{R}^d$. Then it follows from Fatou's lemma that

$$\liminf_{j\to\infty} H_{f_j,\omega,p}(\xi) \geq H_{f,\omega,p}(\xi) \quad \text{and} \quad \liminf_{j\to\infty} \|f_j\|_{M_{(\omega)}^{p,q}} \geq \|f\|_{M_{(\omega)}^{p,q}}.$$

The following proposition is important to us later on. We omit the proof since the result is a restatement of Proposition 2.3 in [39].

Proposition 1.5. *Assume that* $p, q \in [1, \infty]$ *with* $q < \infty$ *and that* $\omega \in \mathscr{P}(\mathbf{R}^{2d})$. *Then* $C_0^\infty(\mathbf{R}^d)$ *is dense in* $M_{(\omega)}^{p,q}(\mathbf{R}^d)$ *with respect to the narrow convergence.*

Next we recall some facts in Chapter XVIII in [23] concerning pseudo-differential operators. Assume that $a \in \mathscr{S}(\mathbf{R}^{2d})$, and that $t \in \mathbf{R}$ is fixed. Then the pseudo-differential operator $a_t(x, D)$ in

(1.4)
$$(a_t(x, D)f)(x) = (\mathrm{Op}_t(a)f)(x)$$
$$= (2\pi)^{-d} \iint a((1-t)x + ty, \xi)f(y)e^{i\langle x-y, \xi\rangle} \, dy d\xi.$$

is a linear and continuous operator on $\mathscr{S}(\mathbf{R}^d)$. For general $a \in \mathscr{S}'(\mathbf{R}^{2d})$, the pseudo-differential operator $a_t(x, D)$ is defined as the continuous operator from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ with distribution kernel

(1.5) $\quad K_{t,a}(x, y) = (2\pi)^{-n/2}(\mathscr{F}_2^{-1}a)((1-t)x + ty, y - x),$

where $\mathscr{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the y-variable. This definition makes sense, since the mappings $\mathscr{F}_2$ and $F(x, y) \mapsto F((1-t)x + ty, y - x)$ are homeomorphisms on $\mathscr{S}'(\mathbf{R}^{2d})$. Moreover, it agrees with the operator in (1.4) when $a \in$

$\mathscr{S}(\mathbf{R}^{2d})$. If $t = 0$, then $a_t(x, D)$ agrees with the Kohn-Nirenberg representation $a(x, D)$. If instead $t = 1/2$, then $a_t(x, D)$ is the Weyl operator $a^w(x, D)$ of a, and if $a, b \in \mathscr{S}(\mathbf{R}^{2d})$, then the Weyl product $a\#b$ between a and b is the function which fulfills $(a\#b)^w(x, D) = a^w(x, D)b^w(x, D)$.

Next we recall the definition of symplectic Fourier transform, twisted convolution and related objects. The even-dimensional vector space $\mathbf{R}^{2d}$ is a (real) symplectic vector space with the (standard) symplectic form

$$\sigma(X, Y) = \sigma\big((x, \xi); (y, \eta)\big) = \langle y, \xi \rangle - \langle x, \eta \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on $\mathbf{R}^d$.

The symplectic Fourier transform for $a \in \mathscr{S}(\mathbf{R}^{2d})$ is defined by the formula

$$(\mathscr{F}_\sigma a)(X) = \widehat{a}(X) = \pi^{-d} \int a(Y) e^{2i\sigma(X,Y)} \, dY.$$

Then $\mathscr{F}_\sigma^{-1} = \mathscr{F}_\sigma$ is continuous on $\mathscr{S}(\mathbf{R}^{2d})$, and extends as usual to a homeomorphism on $\mathscr{S}'(\mathbf{R}^{2d})$, and to a unitary map on $L^2(\mathbf{R}^{2d})$. The symplectic short-time Fourier transform of $a \in \mathscr{S}'(\mathbf{R}^{2d})$ with respect to the window function $\varphi \in \mathscr{S}'(\mathbf{R}^{2d})$ is defined by

$$\mathcal{V}_\varphi a(X, Y) = \mathscr{F}_\sigma\big(a\,\varphi(\,\cdot\, - X)\big)(Y), \quad X, Y \in \mathbf{R}^{2d}.$$

Assume that $\omega \in \mathscr{P}(\mathbf{R}^{4d})$. Then we let $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $\mathcal{W}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ denote the modulation spaces, where the symplectic short-time Fourier transform is used instead of the usual short-time Fourier transform in the definitions of the norms. It follows that any property valid for $M_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ or $W_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ carry over to $\mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ and $\mathcal{W}_{(\omega)}^{p,q}(\mathbf{R}^{2d})$ respectively. For example, for the symplectic short-time Fourier transform we have

(1.6) $$\mathcal{V}_{\mathscr{F}_\sigma \varphi}(\mathscr{F}_\sigma a)(X, Y) = e^{2i\sigma(Y,X)} \mathcal{V}_\varphi a(Y, X),$$

which implies that

(1.7) $$\mathscr{F}_\sigma \mathcal{M}_{(\omega)}^{p,q}(\mathbf{R}^{2d}) = \mathcal{W}_{(\omega_0)}^{q,p}(\mathbf{R}^{2d}), \qquad \omega_0(X, Y) = \omega(Y, X).$$

Assume that $a, b \in \mathscr{S}(\mathbf{R}^{2d})$. Then the twisted convolution of a and b is defined by the formula

(1.8) $$(a *_\sigma b)(X) = (2/\pi)^{d/2} \int a(X - Y) b(Y) e^{2i\sigma(X,Y)} \, dY.$$

The definition of $*_\sigma$ extends in different ways. For example, it extends to a continuous multiplication on $L^p(\mathbf{R}^{2d})$ when $p \in [1, 2]$, and to a continuous map from $\mathscr{S}'(\mathbf{R}^{2d}) \times \mathscr{S}(\mathbf{R}^{2d})$ to $\mathscr{S}'(\mathbf{R}^{2d})$. If $a, b \in \mathscr{S}'(\mathbf{R}^{2d})$, then $a\#b$ makes sense if and only if $a *_\sigma \widehat{b}$ makes sense, and then

(1.9) $$a\#b = (2\pi)^{-d/2} a *_\sigma (\mathscr{F}_\sigma b).$$

We also remark that for the twisted convolution we have

(1.10) $$\mathscr{F}_\sigma(a *_\sigma b) = (\mathscr{F}_\sigma a) *_\sigma b = \check{a} *_\sigma (\mathscr{F}_\sigma b),$$

where $\check{a}(X) = a(-X)$ (cf. [32, 33, 35]). A combination of (1.9) and (1.10) give

$$(1.11) \qquad \mathscr{F}_\sigma(a\#b) = (2\pi)^{-d/2}(\mathscr{F}_\sigma a) *_\sigma (\mathscr{F}_\sigma b).$$

2. Twisted convolution on modulation spaces and Lebesgue spaces

In this section we discuss algebraic properties of the twisted convolution when acting on modulation spaces of the form $W^{p,q}_{(\omega)}$. The most general result is equivalent to Theorem 0.3′ in [22], which concerns continuity for the Weyl product on modulation spaces. Thereafter we use this result to establish continuity properties for the twisted convolution when acting on weighted Lebesgue spaces.

The following lemma is important in our investigations. The proof is omitted since the result is an immediate consequence of Lemma 4.4 in [33] and its proof, (1.6), (1.9) and (1.10).

Lemma 2.1. *Assume that $a_1 \in \mathscr{S}'(\mathbf{R}^{2d})$, $a_2 \in \mathscr{S}(\mathbf{R}^{2d})$ and $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbf{R}^{2d})$. Then the following is true:*

(1) if $\varphi = \pi^d \varphi_1 \# \varphi_2$, then $\varphi \in \mathscr{S}(\mathbf{R}^{2d})$, the map

$$Z \mapsto e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1} a_1)(X - Y + Z, Z)\, (\mathcal{V}_{\chi_2} a_2)(X + Z, Y - Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$(2.1) \quad \mathcal{V}_\varphi(a_1 \# a_2)(X, Y)$$

$$= \int e^{2i\sigma(Z,Y)}(\mathcal{V}_{\chi_1} a_1)(X - Y + Z, Z)\, (\mathcal{V}_{\chi_2} a_2)(X + Z, Y - Z)\, dZ;$$

(2) if $\varphi = 2^{-d} \varphi_1 *_\sigma \varphi_2$, then $\varphi \in \mathscr{S}(\mathbf{R}^{2d})$, the map

$$Z \mapsto e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1} a_1)(X - Y + Z, Z)\, (\mathcal{V}_{\chi_2} a_2)(Y - Z, X + Z)$$

belongs to $L^1(\mathbf{R}^{2d})$, and

$$(2.2) \quad \mathcal{V}_\varphi(a_1 *_\sigma a_2)(X, Y)$$

$$= \int e^{2i\sigma(X,Z-Y)}(\mathcal{V}_{\chi_1} a_1)(X - Y + Z, Z)\, (\mathcal{V}_{\chi_2} a_2)(Y - Z, X + Z)\, dZ$$

The first part of the latter result is used in [22] to prove the following result, which is essentially a restatement of Theorem 0.3′ in [22]. Here we assume that the involved weight functions satisfy

$$(2.3) \quad \omega_0(X, Y) \le C\omega_1(X - Y + Z, Z)\omega_2(X + Z, Y - Z), \quad X, Y, Z \in \mathbf{R}^{2d}.$$

for some constant $C > 0$, and that $p_j, q_j \in [1, \infty]$ satisfy

$$(2.4) \qquad \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} = 1 - \left(\frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0}\right)$$

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and

$$(2.5) \qquad 0 \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0} \leq \frac{1}{p_j}, \frac{1}{q_j} \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0}, \quad j = 0, 1, 2.$$

Theorem 2.2. *Assume that* $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{4d})$ *satisfy* (2.3), *and that* $p_j, q_j \in [1, \infty]$ *for* $j = 0, 1, 2$, *satisfy* (2.4) *and* (2.5). *Then the map* (0.1) *on* $\mathscr{S}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $M^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times M^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $M^{p_0, q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, *and for some constant* $C > 0$, *the bound* (0.2) *holds for every* $a_1 \in M^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in M^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

The next result is an immediate consequence of (1.7), (1.11) and Theorem 2.2. Here the condition (2.3) should be replaced by

$$(2.6) \quad \omega_0(X, Y) \leq C\omega_1(X - Y + Z, Z)\omega_2(Y - Z, X + Z), \quad X, Y, Z \in \mathbf{R}^{2d}.$$

and the condition (2.5) should be replaced by

$$(2.7) \qquad 0 \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q_0} \leq \frac{1}{p_j}, \frac{1}{q_j} \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_0}, \quad j = 0, 1, 2.$$

Theorem 2.3. *Assume that* $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{4d})$ *satisfy* (2.6), *and that* $p_j, q_j \in [1, \infty]$ *for* $j = 0, 1, 2$, *satisfy* (2.4) *and* (2.7). *Then the map* (0.3) *on* $\mathscr{S}(\mathbf{R}^{2d})$ *extends uniquely to a continuous map from* $W^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times W^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $W^{p_0, q_0}_{(\omega_0)}(\mathbf{R}^{2d})$, *and for some constant* $C > 0$, *the bound* (0.4) *holds for every* $a_1 \in W^{p_1, q_1}_{(\omega_1)}(\mathbf{R}^{2d})$ *and* $a_2 \in W^{p_2, q_2}_{(\omega_2)}(\mathbf{R}^{2d})$.

By using Theorem 2.3 we may generalize Proposition 1.4 in [34] to involve continuity of the twisted convolution on weighted Lebesgue spaces.

Theorem 2.4. *Assume that* $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ *and* $p, p_1, p_2 \in [1, \infty]$ *satisfy*

$$\omega_0(X_1 + X_2) \leq C\omega_1(X_1)\omega_2(X_2), \quad p_1, p_2 \leq p$$

$$and \quad \max\left(\frac{1}{p}, \frac{1}{p'}\right) \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \leq 1,$$

for some constant C. *Then the map* (0.3) *extends uniquely to a continuous mapping from* $L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $L^{p}_{(\omega_0)}(\mathbf{R}^{2d})$. *Furthermore, for some constant* C *it holds*

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \leq C\|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}},$$

$$when \quad a_1 \in L^{p_1}_{(\omega_1)}(\mathbf{R}^{2d}), \quad and \quad a_2 \in L^{p_2}_{(\omega_2)}(\mathbf{R}^{2d}).$$

Proof. From the assumptions it follows that at most one of p_1 and p_2 are equal to ∞. By reasons of symmetry we may therefore assume that $p_2 < \infty$.

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Since $W^2_{(\omega)} = M^2_{(\omega)} = L^2_{(\omega)}$ when $\omega(X, Y) = \omega(X)$, in view of Theorem 2.2 in [37], the result follows from Theorem 2.3 in the case $p_1 = p_2 = p = 2$.

Now assume that $1/p_1 + 1/p_2 - 1/p = 1$, $a_1 \in L^{p_1}(\mathbf{R}^{2d})$ and that $a_2 \in \mathscr{S}(\mathbf{R}^{2d})$. Then

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \leq (2/\pi)^{d/2} \| \, |a_1| * |a_2| \, \|_{L^p_{(\omega_0)}} \leq C\|a_1\|_{L^{p_1}_{(\omega_1)}}\|a_2\|_{L^{p_2}_{(\omega_2)}},$$

by Young's inequality. The result now follows in this case from the fact that $\mathscr{S}$ is dense in $L^{p_2}_{(\omega_2)}$, when $p_2 < \infty$.

The result now follows in the general case by multi-linear interpolation between the case $p_1 = p_2 = p = 2$ and the case $1/p_1 + 1/p_2 - 1/p = 1$, using Theorem 4.4.1 in [1] and the fact that

$$(L^{p_1}_{(\omega)}(\mathbf{R}^{2d}), (L^{p_2}_{(\omega)}(\mathbf{R}^{2d}))_{[\theta]} = L^{p_0}_{(\omega)}(\mathbf{R}^{2d}),$$

when

$$\frac{1-\theta}{p_1} + \frac{\theta}{p_2} = \frac{1}{p_0}.$$

(Cf. Chapter 5 in [1].) The proof is complete. $\qquad\square$

By letting $p_1 = p$ and $p_2 = q \leq \min(p, p')$, or $p_2 = p$ and $p_1 = q \leq \min(p, p')$, Theorem 2.4 takes the following form:

Corollary 2.5. *Assume that* $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ *and* $p, q \in [1, \infty]$ *satisfy*

$$\omega_0(X_1 + X_2) \leq C\omega_1(X_1)\omega_2(X_2), \quad and \quad q \leq \min(p, p')$$

for some constant C. Then the map (0.3) *extends uniquely to a continuous mapping from* $L^p_{(\omega_1)}(\mathbf{R}^{2d}) \times L^q_{(\omega_2)}(\mathbf{R}^{2d})$ *or* $L^q_{(\omega_1)}(\mathbf{R}^{2d}) \times L^p_{(\omega_2)}(\mathbf{R}^{2d})$ *to* $L^p_{(\omega_0)}(\mathbf{R}^{2d})$. *Furthermore, for some constant C it holds*

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \leq C\|a_1\|_{L^p_{(\omega_1)}}\|a_2\|_{L^q_{(\omega_2)}}, \; a_1 \in L^p_{(\omega_1)}(\mathbf{R}^{2d}), \; a_2 \in L^q_{(\omega_2)}(\mathbf{R}^{2d})$$

and

$$\|a_1 *_\sigma a_2\|_{L^p_{(\omega_0)}} \leq C\|a_1\|_{L^q_{(\omega_1)}}\|a_2\|_{L^p_{(\omega_2)}}, \; a_1 \in L^q_{(\omega_1)}(\mathbf{R}^{2d}), \; a_2 \in L^p_{(\omega_2)}(\mathbf{R}^{2d}).$$

In the next section we need the following refinement of Theorem 2.4 concerning mixed Lebesgue spaces.

Theorem 2.4′. *Assume that* $k \in \{1, 2\}$, $\omega_0, \omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ *and* $p, p_j, q, q_j \in [1, \infty]$ *for $j = 1, 2$ satisfy*

$$\omega_0(X_1 + X_2) \leq C\omega_1(X_1)\omega_2(X_2), \quad p_1, p_2 \leq p, \quad q_1, q_2 \leq q,$$

$$\max\left(\frac{1}{p}, \frac{1}{p'}, \frac{1}{q}, \frac{1}{q'}\right) \leq \frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p} \leq 1 \quad and$$

$$\max\left(\frac{1}{p}, \frac{1}{p'}, \frac{1}{q}, \frac{1}{q'}\right) \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{q} \leq 1,$$

for some constant C. Then the map (0.3) extends uniquely to a continuous mapping from $L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d}) \times L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d})$ to $L^{p,q}_{k,(\omega_0)}(\mathbf{R}^{2d})$. Furthermore, for some constant C it holds

$$\|a_1 *_\sigma a_2\|_{L^{p,q}_{k,(\omega_0)}} \le C \|a_1\|_{L^{p_1,q_1}_{k,(\omega_1)}} \|a_2\|_{L^{p_2,q_2}_{k,(\omega_2)}},$$

$$when \quad a_1 \in L^{p_1,q_1}_{k,(\omega_1)}(\mathbf{R}^{2d}), \quad and \quad a_2 \in L^{p_2,q_2}_{k,(\omega_2)}(\mathbf{R}^{2d}).$$

Proof. The result follows from Minkowski's inequality when $p_1 = q_1 = 1$ and when $p_2 = q_2 = 1$. Furthermore, the result follows in the case $p_1 = p_2 = q_1 = q_2 = 2$ from Theorem 2.4. In the general case, the result follows from these cases and multi-linear interpolation. $\qquad\square$

3. Window functions in modulation space norms

In this section we use the results in the previous section to prove that the class of permitted windows in the modulation space norm can be enlarged. More precisely we have the following.

Proposition 3.1. *Assume that $p, p_0, q, q_0 \in [1, \infty]$ and $\omega, v \in \mathscr{P}(\mathbf{R}^{2d})$ are such that $p_0, q_0 \le \min(p, p', q, q')$, $\check{v} = v$ and ω is v-moderate. Also assume that $f \in \mathscr{S}'(\mathbf{R}^d)$. Then the following is true:*

(1) *if $\varphi \in M^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in M^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\varphi f \in L^{p,q}_{1,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\varphi f\|_{L^{p,q}_{1,(\omega)}}$ defines a norm for $M^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of φ give rise to equivalent norms;*

(2) *if $\varphi \in W^{p_0,q_0}_{(v)}(\mathbf{R}^d) \setminus 0$, then $f \in W^{p,q}_{(\omega)}(\mathbf{R}^d)$ if and only if $V_\varphi f \in L^{p,q}_{2,(\omega)}(\mathbf{R}^{2d})$. Furthermore, $\|f\| \equiv \|V_\varphi f\|_{L^{p,q}_{2,(\omega)}}$ defines a norm for $W^{p,q}_{(\omega)}(\mathbf{R}^d)$, and different choices of φ give rise to equivalent norms.*

For the proof we recall that the (cross) Wigner distribution of $f \in \mathscr{S}'(\mathbf{R}^d)$ and $g \in \mathscr{S}'(\mathbf{R}^d)$ is defined by the formula

$$W_{f,g}(x, \xi) = \mathscr{F}(f(x/2 - \cdot)\overline{g(x/2 + \cdot)})(\xi).$$

If $f, g \in \mathscr{S}(\mathbf{R}^d)$, then $W_{f,g}$ takes the form

$$W_{f,g}(x, \xi) = (2\pi)^{-d/2} \int f(x - y/2)\overline{g(x + y/2)}e^{i\langle y, \xi\rangle}\, dy.$$

From the definitions it follows that

$$W_{f,g}(x, \xi) = 2^d e^{i\langle x, \xi\rangle/2} V_{\check{g}}f(2x, 2\xi),$$

which implies that

$$(3.1) \quad \|W_{f,\check{\varphi}}\|_{L^{p,q}_{k,(\omega_0)}} = 2^d \|V_\varphi f\|_{L^{p,q}_{k,(\omega)}}, \quad when \quad \omega_0(x, \xi) = \omega(2x, 2\xi)$$

for $k = 1, 2$.

Finally, by Fourier's inversion formula it follows that if $f_1, g_2 \in \mathscr{S}'(\mathbf{R}^d)$ and $f_1, g_2 \in L^2(\mathbf{R}^d)$, then

$$(3.2) \qquad W_{f_1,g_1} *_\sigma W_{f_2,g_2} = (\check{f}_2, g_1)_{L^2} W_{f_1,g_2}.$$

Proof of Theorem 3.1. We may assume that $p_0 = q_0 = \min(p, p', q, q')$. Assume that $\varphi, \psi \in M_{(v)}^{p_0,q_0}(\mathbf{R}^d) \subseteq L^2(\mathbf{R}^d)$, where the inclusion follows from the fact that $p_0, q_0 \leq 2$ and $v \geq c$ for some constant $c > 0$. Since ω is both v-moderate and $\check{v}$-moderate, and $\|V_\varphi \psi\|_{L_{k,(v)}^{p_0,q_0}} = \|V_\psi \varphi\|_{L_{k,(v)}^{p_0,q_0}}$ when $\check{v} = v$, the result follows if we prove that

$$(3.3) \qquad \|V_\varphi f\|_{L_{k,(\omega)}^{p,q}} \leq C \|V_\psi f\|_{L_{k,(\omega)}^{p,q}} \|V_\varphi \psi\|_{L_{k,(v)}^{p_0,q_0}},$$

for some constant C which is independent of $f \in \mathscr{S}'(\mathbf{R}^d)$ and $\varphi, \psi \in M_{(v)}^{p_0,q_0}(\mathbf{R}^d)$.

If $p_1 = p$, $p_2 = p_0$, $q_1 = q$, $q_2 = q_0$, $\omega_0 = \omega(2\cdot)$ and $v_0 = v(2\cdot)$, then Theorem 2.4′ and (3.2) give

$$\|V_\varphi f\|_{L_{k,(\omega)}^{p,q}} = C_1 \|W_{f,\check{\varphi}}\|_{L_{k,(\omega_0)}^{p,q}}$$

$$= C_2 \|W_{f,\check{\psi}} *_\sigma W_{\psi,\check{\varphi}}\|_{L_{k,(\omega_0)}^{p,q}} \leq C_3 \|W_{f,\check{\psi}}\|_{L_{k,(\omega_0)}^{p,q}} \|W_{\psi,\check{\varphi}}\|_{L_{k,(v_0)}^{p_0,q_0}}$$

$$= C_4 \|V_\psi f\|_{L_{k,(\omega)}^{p,q}} \|V_\varphi \psi\|_{L_{k,(v)}^{p_0,q_0}},$$

and (3.3) follows. The proof is complete. $\qquad \square$

4. Pseudo-differential operators with symbols in modulation spaces

In this section we discuss continuity of pseudo-differential operators with symbols in modulation spaces of the form $W_{(\omega)}^{p,q}$, when acting between modulation spaces.

The main result is the following theorem, which is based on Proposition 4.2.

Theorem 4.1. *Assume that $p, q \in [1, \infty]$, $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$ and $\omega \in \mathscr{P}(\mathbf{R}^{4d})$ are such that*

$$(4.1) \qquad \frac{\omega_2(x, \xi + \eta)}{\omega_1(x + y, \xi)} \leq C\omega(x, \xi, \eta, y)$$

holds for some constant C which is independent of $x, y, \xi, \eta \in \mathbf{R}^d$. Also assume that $a \in W_{(\omega)}^{q,p}(\mathbf{R}^{2d})$. Then the definition of $a(x, D)$ from $\mathscr{S}(\mathbf{R}^d)$ to $\mathscr{S}'(\mathbf{R}^d)$ extends uniquely to a continuous mapping from $M_{(\omega_1)}^{p',q'}(\mathbf{R}^d)$ to $W_{(\omega_2)}^{q,p}(\mathbf{R}^d)$. Furthermore, it holds

$$\|a(x, D)f\|_{W_{(\omega_2)}^{q,p}} \leq C\|a\|_{W_{(\omega)}^{q,p}} \|f\|_{M_{(\omega_1)}^{p',q'}}$$

for some constant C which is independent of $f \in M_{(\omega_1)}^{p',q'}(\mathbf{R}^d)$ and $a \in W_{(\omega)}^{q,p}(\mathbf{R}^{2d})$.

The proof is based on duality, using the fact that if $a \in \mathscr{S}'(\mathbf{R}^{2d})$, $f, g \in \mathscr{S}(\mathbf{R}^d)$ and T is the operator, defined by

$$(4.2) \qquad (T\psi)(x, \xi) = \psi(\xi, -x)$$

when $\psi \in \mathscr{S}'(\mathbf{R}^{2d})$, then

$$(4.3) \qquad (a(x, D)f, g)_{L^2(\mathbf{R}^d)} = (2\pi)^{-d/2}(T(\mathscr{F}a), V_f g)_{L^2(\mathbf{R}^{2d})},$$

by Fourier's inversion formula.

For the proof of Theorem 4.1 we shall combine (4.3) with the following weighted version of Proposition 3.3 in [2].

Proposition 4.2. *Assume that $f_1, f_2 \in \mathscr{S}'(\mathbf{R}^d)$, $p, q \in [1, \infty]$, $\omega_0 \in \mathscr{P}(\mathbf{R}^{4d})$ and $\omega_1, \omega_2 \in \mathscr{P}(\mathbf{R}^{2d})$. Also assume that $\varphi_1, \varphi_2 \in \mathscr{S}(\mathbf{R}^d)$, and let $\Psi = V_{\varphi_1}\varphi_2$. Then the following is true:*

(1) *if*

$$(4.4) \qquad \omega_0(x, \xi, \eta, y) \le C\omega_1(-x - y, \eta)\omega_2(-y, \xi + \eta)$$

for some constant C, then

$$\|V_\Psi(V_{f_1}f_2)\|_{L^{p,q}_{1,(\omega_0)}} \le C\|V_{\varphi_1}f_1\|_{L^{p,q}_{1,(\omega_1)}}\|V_{\varphi_2}f_2\|_{L^{q,p}_{2,(\omega_2)}};$$

(2) *if*

$$(4.5) \qquad \omega_1(-x - y, \eta)\omega_2(-y, \xi + \eta) \le C\omega_0(x, \xi, \eta, y)$$

for some constant C, then

$$\|V_{\varphi_1}f_1\|_{L^{p,q}_{1,(\omega_1)}}\|V_{\varphi_2}f_2\|_{L^{q,p}_{2,(\omega_2)}} \le C\|V_\Psi(V_{f_1}f_2)\|_{L^{p,q}_{1,(\omega_0)}};$$

(3) *if (4.4) and (4.5) hold for some constant C, then $f_1 \in M^{p,q}_{(\omega_1)}(\mathbf{R}^d)$ and $f_2 \in W^{q,p}_{(\omega_2)}(\mathbf{R}^d)$, if and only if $V_{f_1}f_2 \in M^{p,q}_{(\omega_0)}(\mathbf{R}^{2d})$, and*

$$C^{-1}\|V_{f_1}f_2\|_{M^{p,q}_{(\omega_0)}} \le \|f_1\|_{M^{p,q}_{(\omega_1)}}\|f_2\|_{W^{q,p}_{(\omega_2)}} \le C\|V_{f_1}f_2\|_{M^{p,q}_{(\omega_0)}},$$

for some constant C which is independent of f_1 and f_2.

Proof. It suffices to prove (1) and (2), and then we prove only (1), since (2) follows by similar arguments. We shall mainly follow the proof of Proposition 3.3 in [2], and then we only prove the result in the case $p < \infty$ and $q < \infty$. The small modifications when $p = \infty$ or $q = \infty$ are left for the reader.

By Fourier's inversion formula we have

$$|V_\varphi f_1(-x - y, \eta)V_\varphi f_2(-y, \xi + \eta)| = |V_\Psi(V_{f_1}f_2)(x, \xi, \eta, y)|$$

(cf. e. g. [2, 12, 15, 31, 32]). Hence, if

$$F_1(x, \xi) = V_{\varphi_1}f_1(x, \xi)\omega_1(x, \xi) \quad \text{and} \quad F_2(x, \xi) = V_{\varphi_2}f_2(x, \xi)\omega_2(x, \xi),$$

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then we get

$$\|V_\Psi(V_{f_1}f_2)\|^q_{L^{p,q}_{1,(\omega_0)}}$$

$$= \iint_{\mathbf{R}^{2d}} \left(\iint_{\mathbf{R}^{2d}} |V_\Psi(V_{f_1}f_2)(x,\xi,\eta,y)\omega_0(x,\xi,\eta,y)|^p \, dxd\xi \right)^{q/p} dyd\eta$$

$$\leq C^q \iint_{\mathbf{R}^{2d}} \left(\iint_{\mathbf{R}^{2d}} |F_1(-x-y,\eta)F_2(-y,\xi+\eta))|^p \, dxd\xi \right)^{q/p} dyd\eta.$$

By taking $-y$, $\xi+\eta$, $-x-y$ and η as new variables of integration, we obtain

$$\|V_\Psi(V_{f_1}f_2)\|^q_{L^{p,q}_{1,(\omega_0)}}$$

$$\leq C^q \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F_1(x,\eta)|^p \, dx \right)^{q/p} d\eta \right) \left(\int_{\mathbf{R}^d} \left(\int_{\mathbf{R}^d} |F_2(y,\xi)|^p \, d\xi \right)^{q/p} dy \right)$$

$$= C^q \|V_{\varphi_1}f_1\|^q_{L^{p,q}_{1,(\omega_1)}} \|V_{\varphi_2}f_2\|^q_{L^{q,p}_{2,(\omega_2)}}.$$

This proves the assertion. □

Proof of Theorem 4.1. We may assume that (4.1) holds for $C = 1$ and with equality. We start to prove the result when $1 < p$ and $1 < q$. Let

$$\omega_0(x,\xi,\eta,y) = \omega(-y,\eta,\xi,-x)^{-1},$$

and assume that $a \in W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$ and $f,g \in \mathscr{S}(\mathbf{R}^d)$. Then $a(x,D)f$ makes sense as an element in $\mathscr{S}'(\mathbf{R}^d)$.

By Proposition 4.2 we get

$$(4.6) \qquad \|V_f g\|_{M^{p',q'}_{(\omega_0)}} \leq C\|f\|_{M^{p',q'}_{(\omega_1)}} \|g\|_{W^{q',p'}_{(\omega_2^{-1})}}.$$

Furthermore, if T is the same as in (4.2), then it follows by Fourier's inversion formula that

$$(V_\varphi(T\widehat{a}))(x,\xi,\eta,y) = e^{-i(\langle x,\eta\rangle + \langle y,\xi\rangle)}(V_{T\widehat{\varphi}}a)(-y,\eta,\xi,-x).$$

This gives

$$|(V_\varphi(T\widehat{a}))(x,\xi,\eta,y)\omega_0(x,\xi,\eta,y)^{-1}|$$

$$= |(V_{\varphi_1}a)(-y,\eta,\xi,-x)\omega(-y,\eta,\xi,-x)|,$$

when $\varphi_1 = T\widehat{\varphi}$. Hence, by applying the $L^{p,q}_1$ norm we obtain $\|T\widehat{a}\|_{M^{p,q}_{(\omega_0^{-1})}} = \|a\|_{W^{q,p}_{(\omega)}}$.

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It now follows from (12) and (4.6) that

$$(4.7) \quad |(a(x,D)f, g)| = (2\pi)^{-d/2}|(T\widehat{a}, V_{\overline{g}}\overline{f})|$$

$$\leq C_1 \|T\widehat{a}\|_{M^{p,q}_{(\omega_0^{-1})}} \|V_f g\|_{M^{p',q'}_{(\omega_0)}}$$

$$\leq C_2 \|a\|_{W^{q,p}_{(\omega)}} \|f\|_{M^{p',q'}_{(\omega_1)}} \|g\|_{W^{q',p'}_{(\omega_2^{-1})}}.$$

The result now follows by the facts that $\mathscr{S}(\mathbf{R}^d)$ is dense in $M^{p',q'}_{(\omega_1)}(\mathbf{R}^d)$, and that the dual of $W^{q',p'}_{(\omega_2^{-1})}$ is $W^{q,p}_{(\omega_2)}$ when $p, q > 1$.

If instead $p = 1$ and $q < \infty$, or $q = 1$ and $p < \infty$, then we assume that $f \in M^{p',q'}_{(\omega_1)}$ and $a \in \mathscr{S}(\mathbf{R}^{2d})$. Then $a(x,D)f$ makes sense as an element in $\mathscr{S}(\mathbf{R}^d)$, and from the first part of the proof it follows that (4.7) still holds. The result now follows by duality and the fact that $\mathscr{S}(\mathbf{R}^{2d})$ is dense in $W^{q,p}_{(\omega)}(\mathbf{R}^{2d})$ for such choices of p and q.

It remains to consider the cases $p = q' = 1$ and $p = q' = \infty$. In this case, the result follows by using the fact that $\mathscr{S}$ is dense in $M^{\infty,1}_{(\omega)}$ and $W^{1,\infty}_{(\omega)}$ with respect to the narrow convergence. The proof is complete. $\qquad\qquad\square$

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